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Indian Statistical Institute
B.Math.(Hons.) II Year
First Semester Examination, 2005-2006
Algebra III

Time: 3 hrs Date: 21-11-05 Total Marks : 50

Attempt all questions

1. State whether the following statements are true or false and justify your answer (quote a relevent theorem or give an argument/counterexample)

   a) If L/K and K/F are Galois extensions then so is L/F.

   b) A regular 17-gon is constructible by ruler and compass.

   c) Any finite group is the Galois group of an extension K/F of fields for some K and F.

   d) Let K be the splitting field of $x^3 + x + 1$ over $\mathbb{Q}$. Then $\sqrt{-1} \in K$.

   (5×4)

2. a) Let p be a prime number, $\mathbb{F}_p$ the finite field with p elements and $\mathbb{F}_q$, for $q = p^n$, be the extension of $\mathbb{F}_p$ of degree n. Consider the Frobenius map $\varphi : \mathbb{F}_q \to \mathbb{F}_q$ defined by $\varphi(\alpha) = \alpha^p$ for $\alpha \in \mathbb{F}_q$. Prove that φ is a field automorphism of $\mathbb{F}_q$ which fixes $\mathbb{F}_p$. Show that the Galois group $G(\mathbb{F}_q/\mathbb{F}_p)$ is cyclic and is generated by φ.

   b) Let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Prove that the Galois group $G(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is an abelian group. (5+5)

3. Are the roots of the following equations expressible by radicals over $\mathbb{Q}$? Justify your answer.

   a) $x^5 - 4x + 2$

   b) $x^5 - 2$ (5+5)

4. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 2x^2 - 1$. Find the Galois group G of $K/\mathbb{Q}$, explicitly determine the action of G on the set of roots. Find all intermediate fields of $K/\mathbb{Q}$ and match them up with the corresponding subgroups of G. Identify all the subextensions of $K/\mathbb{Q}$ which are Galois. [10]